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Integrability and strong normal forms for non-autonomous systems in a neighbourhood of an equilibrium

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The paper deals with the problem of existence of a convergent "strong" normal form in the neighbourhood of an equilibrium, for a finite dimensional system of differential equations with analytic and time-dependent non-linear terms. The problem can be solved either under some *non-resonance* hypotheses on the spectrum of the linear part or if the non-linear term is assumed to be (slowly) decaying in time. This paper "completes" a pioneering work of Pustyl'nikov in which, despite under weaker non-resonance hypotheses, the nonlinearity is required to be asymptotically autonomous. The result is obtained as a consequence of the existence of a strong normal form for a suitable class of real-analytic Hamiltonians with non-autonomous perturbations. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4962802]

I. PRELIMINARIES AND MAIN RESULT

A. Introduction

The study of the dynamics of a system of ordinary differential equations (ODEs) in a neighbourhood of an equilibrium boasts nowadays a rich and well established theory. Its foundation goes even back to the late XIX century to the contribution of Poincaré[14] and Lyapunov[12]. Given an analytic vector field, the possibility to write the motions of the associated system in the vicinity of an equilibrium as a convergent power series is deeply related to some *non-resonance* conditions on the eigenvalues of the linear part.

The results have been afterwards extended in the studies of Siegel started in Ref. 16. The problem of the reducibility of a given system to a linear form via an analytic transformation, it is shown to be solvable in Ref. 17 for a full measure set of eigenvalues.

In the case of Hamiltonian structure, investigated later in Ref. 18, the problem can be naturally interpreted in terms of the existence of a (convergent) canonical transformation of variables, casting a Hamiltonian of the form "quadratic" + "perturbation" into a suitable normal form (i.e., such that the corresponding canonical equations are integrable), in some neighbourhood of the examined equilibrium. Based on this approach, the paper[10] provides a generalisation of the results by Lyapunov, removing the hypothesis of purely imaginary eigenvalues.

In any case, we remark that, as a common feature of this class of problems, without any assumption on the eigenvalues, the program of casting the Hamiltonian at hand into a normal form, at least in general, fails. In fact, it is immediate to recognize how the linear combinations of eigenvalues occurring in the normalization scheme could produce some "small divisor effects." Knowingly, this phenomenon can either obstruct the formal resolvability of the homological equations produced during the normalization or jeopardize the convergence of the series.

We recall that, for instance, the described problem of well-posedness of the homological equation is overcome by Moser in Ref. 13, in the case of "one and a half" degrees of freedom Hamiltonian $H(p,q,t)$ (with periodic time dependence) close to a hyperbolic equilibrium located at $p = q = 0$. The strategy consists of keeping terms of the form $(pq)^k$, $k \geq 2$, in the normal form. In this way the canonical equations are still integrable ($x := pq$ is a prime integral) but this allows to avoid the division by zero in the homological equation which would have been carried by those terms. This analysis plays a fundamental role in the context of instability phenomena in Hamiltonian systems with several degrees of freedom (Arnold's diffusion), in order to describe the flow in the neighbourhood of partially hyperbolic tori of *a priori* unstable systems, see Ref. 3.

The pioneering work by Pustyl'nikov[15] aims to extend the results of the paper[17] by introducing a time dependence in the non-linear part of the vector field (not necessarily Hamiltonian). As it is natural, the choice of a suitable class of time-dependent perturbations and its treatment is a further difficulty to the phenomenon of the "resonances." In Ref. 15, under the non-resonance condition already assumed in Ref. 17 for the autonomous case, it is required that the perturbation is asymptotic to a time-independent, analytic function. However, no restrictions are imposed on the "type" of the time dependence, more specifically, it has to be neither periodic nor quasi-periodic. This case is also known as *aperiodic* time dependence.

After Ref. 15, the interest in a general dependence on time has been renewed in Ref. 11 then followed by Refs. 2 and 5, and subsequent papers. Basically, all of them deal with the Hamiltonian case (see Ref. 6 for the case of Poisson systems). The paper[7] extends the above described result by Moser to the case of a perturbation aperiodically dependent on time.

As a matter of fact, the Hamiltonian structure is not a real obstruction for the use of the tools apt to treat the Hamiltonian case. In fact, given a system of ODEs, it can be always interpreted as ("a half" of the) canonical equations of a suitable Hamiltonian system, of larger dimension, see, e.g., Ref. 1. The strategy of this paper is to derive the integrability of the system of ODEs at hand, see (7), as a particular case of the existence of a normal form for a real-analytic Hamiltonian with aperiodic perturbation, see (1), by using the tools introduced in Ref. 7 for the one degree of freedom case.

The possibility to cast the Hamiltonian (1) into a normal form is shown to be possible in the two cases described in Theorem 1.1. In the second case, we deal with perturbations linear in the y variables, in the presence of some non-resonance assumption on the eigenvalues. This case is directly related to the Hamiltonian formulation of a system of ODEs (due to the linearity in y). It is immediate to notice that, with respect to [Ref. 15, (0.3)], the condition (4) on the eigenvalues is clearly more restrictive. Nevertheless, the hypothesis of asymptotic time-independence assumed in Ref. 15 is weakened to the simple boundedness.

On the other hand, the first case has a more general character: if the perturbation decays in time, either the described assumption on the form of f or on the eigenvalues turn out to be unnecessary. Basically, the presence of resonance phenomena is no longer an obstruction for the existence of the normal form, see also Ref. 8. We recall that in the latter paper, the exponential decay, see (3), is chosen for simplicity of discussion: the only necessary assumption is the summability in t of the perturbing function over the non-negative real semi-axis.

This paper, based on the *Lie series* formalism developed by A. Giorgilli *et al.*, can be regarded, at the same time, as a non-autonomous version of Ref. 10.

B. Setting

Let us consider the following Hamiltonian

$$H(x,y,\eta,t) = h(x,y,\eta) + f(x,y,t), \qquad h(x,y,\eta) := \eta + \sum_{l=1}^{n} \lambda_l x_l y_l, \tag{1}$$

where $(x,y,\eta) \in \mathcal{D} := [-r,r]^n \times [-r,r]^n \times \mathbb{R}$, with $n \geq 1$ and $r > 0$, $\lambda_l \in \mathbb{C}$ and $t \in \mathbb{R}^+ := [0,+\infty)$. The assumptions on f will be discussed below. The system (1) is nothing but the "autonomous equivalent" of $\mathcal{H}(x,y,t) = \sum_{l=1}^{n} \lambda_l x_l y_l + f(y,x,t)$, once η has been defined as the conjugate variable to t.

The standard use of the analytic tools requires the complexification of the domain $\mathcal{D}$ as follows. Given $R \in (0,1/2]$ set $\mathcal{D}_R := \mathcal{Q}_R \times \mathcal{S}_R$, where

$$\mathcal{Q}_R := \{(x,y) \in \mathbb{C}^{2n} : |x|,|y| \leq R\}, \qquad \mathcal{S}_R := \{\eta \in \mathbb{C} : |\Im\eta| \leq R\}.$$

It will be required that, for all $t \in \mathbb{R}^+$, f belongs to the space of real-analytic functions on $\mathring{\mathcal{Q}}_R$ and continuous on the boundary, which we denote with $\mathfrak{C}(\mathcal{Q}_R)$. As a consequence $H \in \mathfrak{C}(\mathcal{D}_R)$.

In particular, the space of all the $G \in \mathfrak{C}(\mathcal{Q}_R)$ is endowed with the *Taylor norm*

$$\|G(x,y,t)\|_R := \sum_{\alpha,\beta\in\mathbb{N}^n} |g_{\alpha,\beta}(t)| R^{|\alpha+\beta|}, \tag{2}$$

where $G(x,y,t) =: \sum_{\alpha,\beta\in\mathbb{N}^n} g_{\alpha,\beta}(t)x^\alpha y^\beta$ and $|\alpha| := \sum_{l=1}^n \alpha_l$ (it is understood that $x^\alpha y^\beta := x_1^{\alpha_1}\cdot\ldots\cdot x_n^{\alpha_n}\cdot y_1^{\beta_1}\cdot\ldots\cdot y_n^{\beta_n}$). We recall the standard result for which, if $G\in\mathfrak{C}(Q_R)$ for all $t\in\mathbb{R}^+$, then $|g_{\alpha,\beta}(t)|\le|G(\cdot,\cdot,t)|_R R^{-|\alpha+\beta|}$, where $|G(\cdot,\cdot,t)|_R := \sup_{(x,y)\in Q_R}|G|$. In particular, $\|G\|_{R'}<+\infty$ for all $R'<R$.

Throughout this paper we shall deal with perturbations satisfying the following conditions:

1. f is "at least" quadratic in x and "at least" linear in y: a property that we will denote with $(QxLy)$, i.e., $f_{\alpha,\beta}(t)=0$ for all $t\in\mathbb{R}^+$ and for all $(\alpha,\beta)\in\mathbb{N}^{2n}\setminus\Gamma$, where $\Gamma:=\{(\alpha,\beta)\in\mathbb{N}^{2n} : |\alpha|\ge 2, |\beta|\ge 1\}$;
2. there exist $M_f\in[1,+\infty)$ and $a\in[0,1)$ such that, for all $(x,y,t)\in Q_R\times\mathbb{R}^+$,

$$\|f(x,y,t)\|_R \le M_f e^{-at}. \tag{3}$$

Note that the interval $a\in[0,1)$ is a compact way to denote either the time decay $a\in(0,1)$ or the boundedness $a=0$. As in our previous papers, we recall that we are interested in the case of small a (slow decay) and the upper bound $a=1$ is set for simplicity. On the other hand, it is easy to realise that the case $a\ge 1$ is straightforward.

C. Main result

In the described setting, the main result can be stated as follows.

Theorem 1.1. *Suppose that one of the following conditions are satisfied:*

I. *Time decay: $a>0$.*
II. *Linearity in y+ non-resonance: $a=0$ and the perturbation is linear in y, denoted by (Ly), i.e., of the form $f(x,y,t)=y\cdot g(x,t)$. In addition, the vector $\Lambda:=(\lambda_1,\ldots,\lambda_n)$ satisfies the* non-resonance condition

$$\max_{l=1,\ldots,n}\left(|\Re\mathcal{U}(\alpha,e_l,\Lambda)|^{-1}\right)\le\gamma|\alpha|^\tau, \qquad \forall\alpha\in\mathbb{N}^n : |\alpha|\ge 2, \tag{4}$$

where $\mathcal{U}(\alpha,\beta,\Lambda):=(\alpha-\beta)\cdot\Lambda$, for some $\gamma>0$ and $\tau\ge n$. e_l stands for the lth vector of the canonical basis of $\mathbb{R}^n$.

Then it is possible to determine R_*, R_0 with $0<R_*<R_0\le R^{16}$ and a family of canonical transformations $(x,y,\eta)=\mathcal{M}(x^{(\infty)},y^{(\infty)},\eta^{(\infty)})$, $\mathcal{M}:\mathcal{D}_{R_*}\to\mathcal{D}_{R_0}$, analytic on $\mathcal{D}_{R_*}$ for all $t\in\mathbb{R}^+$, casting the Hamiltonian (1) into the strong normal form

$$H^{(\infty)}(x^{(\infty)},y^{(\infty)},\eta^{(\infty)}) = h(x^{(\infty)},y^{(\infty)},\eta^{(\infty)}). \tag{5}$$

Remark 1.2. It is immediate to recognize the similarity between (4) and the standard Diophantine condition. Clearly, all the vectors Λ whose real part is a Diophantine vector satisfy condition (4), no matter what the imaginary part is. Hence the set of vectors satisfying (4) is, *a fortiori*, a full-measure set.

As anticipated in the Introduction, we stress that condition (4) is stronger than the non-resonance condition imposed in Ref. 15 and it is not satisfied in the case of purely imaginary Λ.

Remark 1.3. As usually done in the *Lie series method*, see, e.g., Ref. 9, the transformation $\mathcal{M}$ will be constructed as the limit (defined, at the moment, only at a formal level),

$$\mathcal{M} := \lim_{j\to\infty}\mathcal{M}^{(j)}\circ\mathcal{M}^{(j-1)}\circ\ldots\circ\mathcal{M}^{(1)}, \tag{6}$$

where $\mathcal{M}^{(j)} := \exp(\mathcal{L}_{\chi^{(j)}}) \equiv \mathrm{Id}+\sum_{s\ge1}(s!)^{-1}\mathcal{L}^s_{\chi^{(j)}}$ and $\mathcal{L}_{\chi^{(j)}} := \{\cdot,\chi^{(j)}\}$. The *generating sequence* $\{\chi^{(j)}\}_{j\in\mathbb{N}}$, where $\chi^{(j)}=\chi^{(j)}(x,y,t)$, see Ref. 11, is meant to be determined.

We will show (see the proof of Lemma 3.3) that in the case of a perturbation which is (Ly), it is possible to prove that $\chi^{(j+1)}(x,y,t)$ is (Ly) as well, for all $j\in\mathbb{N}$. In such a case, it is easy to check by induction that $x^{(j)}=\mathcal{M}^{(j+1)}x^{(j+1)}$ (recall that the Lie series operator acts component-wise, i.e., $\mathcal{M}^{(j)}x^{(j)}=(\mathcal{M}^{(j)}x_1^{(j)},\ldots,\mathcal{M}^{(j)}x_n^{(j)})$) *does not* depend on the variable $y^{(j+1)}$, for all j.

This is easily checked. As stated above, if $f^{(j-1)}$ is (Ly), then $\chi^{(j)} = y^{(j)} \cdot C^{(j)}(x^{(j)},t)$ for all j. Suppose by induction that $\mathcal{L}^s_{\chi^{(j)}} x^{(j)}$ does not depend on $y^{(j)}$. This is true for $s = 1$ as $\mathcal{L}_{\chi^{(j)}} x^{(j)} = \{x^{(j)}, y^{(j)} \cdot C^{(j)}(x^{(j)},t)\} = \sum_{l=1}^n C^{(j)}_l(x^{(j)},t)$. Hence $\mathcal{L}^{s+1}_{\chi^{(j)}} x^{(j)} = \mathcal{L}_{\chi^{(j)}} \mathcal{L}^s_{\chi^{(j)}} x^{(j)} = \sum_{l=1}^n \partial_{x^{(j)}_l}(\mathcal{L}^s_{\chi^{(j)}} x^{(j)}) C^{(j)}_l$ which does not depend on $y^{(j)}$.

As a consequence, the composition $x \equiv x^{(0)} = \mathcal{M} x^{(\infty)} =: \mathcal{M}_x(x^{(\infty)},t)$ does not depend on $y^{(\infty)}$, i.e., is an analytic map $\mathcal{M}_x : \tilde{Q}_{R_*} \to \tilde{Q}_{R_0}$ parametrised by t, where $\tilde{Q}_R := \{x \in \mathbb{C}^n : |x| \leq R\}$. This will play a key role in Sec. I D.

D. The corollary

Let us consider the following non-linear system:

$$\dot{v} = Av + g(v,t), \tag{7}$$

where $v \in \mathbb{R}^n$, A is a $n \times n$ matrix with real entries and the function g is such that $\partial^\nu_v g(0,t) \equiv 0$ for all $\nu \in \mathbb{N}^n$ such that $|\nu| \leq 1$, i.e., g is at least quadratic in v. We restrict ourselves to the class of diagonalizable A with non-purely imaginary eigenvalues λ_l. In the obvious system of coordinates denoted with x, the system (7) easily reads as

$$\dot{x}_l = \lambda_l x_l + \tilde{g}_l(x,t), \qquad l = 1,\ldots,n. \tag{8}$$

In this framework one can state the next.

Corollary 1.4. Suppose that $f(x,y,t) := y \cdot \tilde{g}(x,t)$ and Λ is such that the conditions described in II of Theorem 1.1 are satisfied. Then the system (8) is integrable in a suitable neighbourhood of the origin.

The same result holds, in particular, without any non-resonance condition on Λ, provided that $\tilde{g}(x,t)$ is such that (3) is satisfied with $a > 0$.

Proof. The key remark, see, e.g., Ref. 1, is that (8) can be interpreted as a set of canonical equations of the Hamiltonian system with Hamiltonian $\mathcal{K} := \eta + \sum_{l=1}^n y_l(\Lambda_l x_l + \tilde{g}_l(x,t))$, i.e., (1) with $f(x,y,t)$ defined in the statement. Hence, by Theorem 1.1, there exists a suitable neighbourhood of the origin endowed with a set of coordinates $(x^{(\infty)}, y^{(\infty)}, \eta^{(\infty)})$, such that $\mathcal{K}$ is cast into the (integrable) strong form $\mathcal{K}^{(\infty)} = \eta^{(\infty)} + \sum_{l=1}^n \lambda_l {y_l}^{(\infty)} {x_l}^{(\infty)}$. Furthermore, as noticed in Remark 1.3, $\mathcal{M}_x$ is an analytic map between x and $x^{(\infty)}$. Hence $x(t) = \mathcal{M}_x(\exp(\mathcal{A}t) x^{(\infty)}(0),t)$, with $\mathcal{A} := \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$, gives the explicit solution of (8). □

II. SOME PRELIMINARY RESULTS

A. Two elementary inequalities

Proposition 2.1. For all $\mathcal{R} \leq e^{-4}$ and all $\delta \leq 1/2$ the following inequalities hold:

$$\sum_{\substack{\nu \in \mathbb{N}^m \\ |\nu| \geq N}} \mathcal{R}^{|\nu|} \leq 2m e^{3m-3} \mathcal{R}^{\frac{3N}{4}}, \qquad \sum_{\nu \in \mathbb{N}^m} |\nu|^\mu (1-\delta)^{|\nu|} \leq C(m,\mu) \delta^{-m-\mu-1}, \tag{9}$$

where $m \geq 2$, $\mu \geq 0$ and $C(m,\mu) := e^{4m+\mu-1}(m+\mu)^{(m+\mu)}/(m-1)!$.

Proof. See Appendix. □

B. A result on the homological equation

Proposition 2.2. Consider the following equation:

$$\mathcal{L}_\chi h + f = 0, \tag{10}$$

where h has been defined in (1) and $f = f(x,y,t) = \sum_{(\alpha,\beta)\in\Gamma} f_{\alpha,\beta}(t) x^\alpha y^\beta$ satisfies $\|f\|_{\tilde{R}} \leq M \exp(-at)$ for some $a \in [0,1)$, $M, \tilde{R} > 0$. The following statements hold for all $\delta \in (0,1/2]$:

1. *If $a > 0$, there exists $C_1 = C_1(n, \Lambda) > 0$ such that*
$$\|\chi\|_{(1-\delta)\tilde{R}}, \|\partial_t \chi\|_{(1-\delta)\tilde{R}} \leq C_1 M a^{-1} \delta^{-2(n+1)}. \tag{11}$$
2. *If $a = 0$, f is of the form $f = y \cdot g(x,t)$ and Λ satisfies (4), there exists $C_2 = C_2(n, \Lambda, \tau, \gamma) > 0$ such that*
$$\|\chi\|_{(1-\delta)\tilde{R}}, \|\partial_t \chi\|_{(1-\delta)\tilde{R}} \leq C_2 M \delta^{-(n+\tau+2)}. \tag{12}$$

Proof. First of all note that $\mathcal{L}_\chi h = \partial_t \chi + \sum_{l=1}^n \lambda_l (x_l \partial_{x_l} - y_l \partial_{y_l}) \chi$. By expanding the generating function as $\chi(x,y,t) = \sum_{(\alpha,\beta)\in\mathbb{N}^{2n}} c_{\alpha,\beta}(t) x^\alpha y^\beta$, Equation (10) reads, in terms of Taylor coefficients, as
$$\dot{c}_{\alpha,\beta}(t) + \mathcal{U}(\alpha,\beta,\Lambda) c_{\alpha,\beta} = f_{\alpha,\beta}(t). \tag{13}$$
The solution of (13) is easily written, for all $(\alpha,\beta) \in \Gamma$, as
$$c_{\alpha,\beta}(t) = e^{-\mathcal{U}(\alpha,\beta,\Lambda)t} \left[c_{\alpha,\beta}(0) + \int_0^t e^{\mathcal{U}(\alpha,\beta,\Lambda)s} f_{\alpha,\beta}(s) ds \right], \tag{14}$$
while trivially $c_{\alpha,\beta}(t) \equiv 0$ for all $(\alpha,\beta) \in \mathbb{N}^{2n} \setminus \Gamma$.

Now denote $\mathcal{U}_R + i\mathcal{U}_I := \mathcal{U}(\alpha,\beta,\Lambda)$ with $\mathcal{U}_{I,R} \in \mathbb{R}$ and recall that, by hypothesis, $|f_{\alpha,\beta}(t)| \leq M\tilde{R}^{-|\alpha+\beta|} e^{-at}$.

Case $a > 0$. For all $(\alpha,\beta) \in \Gamma$ such that $\mathcal{U}_R \geq 0$ we choose $c_{\alpha,\beta}(0) = 0$ then we have
$$|c_{\alpha,\beta}| \leq e^{-\mathcal{U}_R t} \int_0^t e^{\mathcal{U}_R s} |f_{\alpha,\beta}(s)| ds \leq M\tilde{R}^{-|\alpha+\beta|} \int_0^t e^{-as} ds \leq M\tilde{R}^{-|\alpha+\beta|} a^{-1}.$$
Otherwise, for those α and β such that $\mathcal{U}_R < 0$, redefine $\mathcal{U}_R := -\mathcal{U}_R$ with $\mathcal{U}_R > 0$ and choose $c_{\alpha,\beta}(0) := -\int_{\mathbb{R}^+} \exp(\mathcal{U}(\alpha,\beta,\Lambda)s) f_{\alpha,\beta}(s) ds$. Note that $|c_{\alpha,\beta}(0)| < +\infty$. In this case we have $|c_{\alpha,\beta}| \leq \exp(\mathcal{U}_R t) \int_t^\infty \exp(-\mathcal{U}_R s) |f_{\alpha,\beta}(s)| ds \leq M\tilde{R}^{-|\alpha+\beta|} a^{-1}$. Hence $|c_{\alpha,\beta}| \leq M\tilde{R}^{-|\alpha+\beta|} a^{-1}$ for all $(\alpha,\beta) \in \Gamma$. By recalling (2) one gets $\|\chi\|_{(1-\delta)\tilde{R}} \leq Ma^{-1} \sum_{(\alpha,\beta)\in\mathbb{N}^{2n}} (1-\delta)^{|\alpha+\beta|}$. The use of the second of (9) with $\nu := (\alpha,\beta)$ yields the first part of (11) with C_1 set for the moment to $\hat{C}_1 := C(2n, 0)$.

Directly from (13) we get $|\dot{c}_{\alpha,\beta}| \leq |\alpha+\beta| |\Lambda| |c_{\alpha,\beta}| + |f_{\alpha,\beta}| \leq a^{-1} M (1 + |\Lambda|) |\alpha+\beta| \tilde{R}^{-|\alpha+\beta|}$. By (9) with $\mu = 1$ we get the second of part of (11). The constant is chosen as $C_1 := (1 + |\Lambda|) C(2n, 1) > \hat{C}_1$.

Case $a = 0$. In such case, the homological equation reads as
$$\dot{c}_{\alpha,l}(t) + \mathcal{U}(\alpha, e_l, \Lambda) c_{\alpha,l} = f_{\alpha,l}(t), \tag{15}$$
where $f_{\alpha,l} := f_{\alpha,\beta}|_{\beta=e_l}$ (the same notation for $c_{\alpha,l}$), for all $\alpha \in \mathbb{N}^n$ such that $|\alpha| \geq 2$ and for all $l = 1, \ldots, n$. By hypothesis (4), $\mathcal{U}_R \neq 0$. Similarly to the case $a > 0$, if $\mathcal{U}_R > 0$ we set $c_{\alpha,l}(0) = 0$, otherwise, $c_{\alpha,l}(0) := -\int_{\mathbb{R}^+} \exp(\mathcal{U}(\alpha, e_l, \Lambda)s) f_{\alpha,l}(s) ds$. Proceeding as before, one obtains, by using (4),
$$|c_{\alpha,l}(t)| \leq M\mathcal{U}_R^{-1} \tilde{R}^{-|\alpha|-1} \leq \gamma M_j |\alpha|^\tau \tilde{R}^{-|\alpha|-1}.$$
This implies $\|\chi\|_{(1-\delta)\tilde{R}} \leq n\gamma M \sum_{\alpha\in\mathbb{N}^n} |\alpha|^\tau (1-\delta)^{|\alpha|}$ which is, by (9), the first part of (12) with $\hat{C}_2 = n\gamma C(n, \tau)$. On the other hand, from the homological equation, we get $|\dot{c}_{\alpha,l}(t)| \leq M|\alpha|^{\tau+1}(1 + \gamma|\Lambda|)\tilde{R}^{-|\alpha|-1}$. Similarly, the latter yields the second part of (12) with $C_2 := \max\{n(1 + \gamma|\Lambda|)C(n, \tau + 1), \hat{C}_2\}$. □

C. A bound on the Lie operator

Proposition 2.3. Let F, G be two functions such that $\|F\|_{(1-\tilde{d})\tilde{R}}, \|G\|_{(1-\tilde{d})\tilde{R}} < +\infty$ for some $\tilde{d} \in (0, 1/4]$ and $\tilde{R} > 0$. Then for all $s \in \mathbb{N}$ the following bound holds:
$$\left\| \mathcal{L}_G^s F \right\|_{(1-2\tilde{d})\tilde{R}} \leq e^{-2} s! [e^2 (\tilde{R}\tilde{d})^{-2} \|G\|_{(1-\tilde{d})\tilde{R}}]^s \|F\|_{(1-\tilde{d})\tilde{R}}. \tag{16}$$

Proof. Straightforward from [Ref. 10, Sec. 3.2] and [Ref. 9, Lemma 4.2]. □

III. PROOF OF THE MAIN RESULT: CONVERGENCE OF THE NORMAL FORM

A. Preparation of the domains

Taking into account the domain restriction imposed by Proposition 2.3, the canonical transformations will be constructed of the form $\mathcal{M}_{j+1} : \mathcal{D}_{R_{j+1}} \to \mathcal{D}_{R_j} \ni (x^{(j)}, y^{(j)}, \eta^{(j)})$ (understood $(x^{(0)}, y^{(0)}, \eta^{(0)}) \equiv (x, y, \eta)$), where $\{\mathcal{D}_{R_j}\}_{j\in\mathbb{N}}$ is a suitable sequence of nested domains. The sequence $\{R_j\}$ defined in this section will provide, in Lemma 3.3, the appropriate domain restrictions for the convergence of the scheme. Another sequence $\{\epsilon_j\}$, constructed here, will be used to control the size of the remainder.

Lemma 3.1. Suppose that $\epsilon_0, R_0, a, K, \sigma > 0$ are given real numbers and that the following condition holds:

$$\epsilon_0 \leq \epsilon_a := a(2\pi)^{-\sigma} K^{-1}. \tag{17}$$

Then it is possible to construct a sequence $\{d_j\}_{j\in\mathbb{N}}$, with $d_j \in (0, 1/4]$ such that following sequences

$$\epsilon_{j+1} = K a^{-1} d_j^{-\sigma} \epsilon_j^2, \qquad R_{j+1} := (1 - 2d_j) R_j \tag{18}$$

satisfy $\epsilon_j < 1$, $1 > R_j \geq R_* := R_0/2$ and $\epsilon_j \to 0$ monotonically as $j \to \infty$.

Remark 3.2. The property $R_* > 0$ is crucial, as R_* is the lower bound for the analyticity radius of the normalised Hamiltonian.

Proof. Straightforward from [Ref. 8, Lemma 4.4]. We recall that a suitable choice is $\epsilon_j = \epsilon_0 (j+1)^{-\sigma}$, then, by (18), $d_j = (\epsilon_0 K a^{-1})^{(1/\sigma)} (j+2)^2/(j+1)^4$. From the latter, one has

$$\sum_{j\geq 0} d_j \leq 1/6, \tag{19}$$

provided that condition (17) is satisfied. □

B. Iterative lemma

Let us define for all $j \geq 1$, $H^{(j+1)} := \mathcal{M}^{(j+1)} H^{(j)}$ with $H^{(0)} := H$.

Lemma 3.3. Under the same hypotheses of Theorem 1.1 and under the condition (17) it is possible to find R_0 and a sequence $\{\chi^{(j)}\}_{j\in\mathbb{N}}$ such that $H^{(j)}(x, y, \eta, t) = h(x, y, \eta) + f^{(j)}(x, y, t)$ with $f^{(j)}$ $(QxLy)$ and such that $\|f^{(j)}\|_{R_j} \leq \epsilon_j e^{-at}$ for all j, where ϵ_j, R_j are given by (18).

The stated result exploits the possibility to remove the perturbation with the normalization algorithm obtaining, in this way, the desired normal form (5). We will construct $\mathcal{M}^{(j+1)}$ as the Lie series generated by $\chi^{(j+1)}$ (see definition after (6)), then showing the convergence of the composition (6).

The interpretation of ϵ_j as a bound for the remainder is clearly related to the well-known feature of the *quadratic method*.

Proof. By induction. If $j = 0$, the statement is clearly true by hypothesis, by setting $f^{(0)} := f$, either in the case I or in the case II. We are supposing here that ϵ_0 is small enough in order to satisfy (17). This will be achieved later by a suitable choice of R_0.

Let us suppose the statement to be valid for j. In this way we get

$$H^{(j+1)} \equiv \exp(\mathcal{L}_{\chi^{(j+1)}}) H^{(j)} = h + f^{(j)} + \mathcal{L}_{\chi^{(j+1)}} h + \sum_{s\geq 1} (s!)^{-1} \mathcal{L}^s_{\chi^{(j+1)}} f^{(j)} + \sum_{s\geq 2} (s!)^{-1} \mathcal{L}^s_{\chi^{(j+1)}} h.$$

We shall determine $\chi^{(j+1)}$ in such a way (10) is satisfied so that, by setting

$$f^{(j+1)} := \sum_{s\geq 1} \frac{1}{s!} \mathcal{L}^s_{\chi^{(j+1)}} f^{(j)} + \sum_{s\geq 2} \frac{1}{s!} \mathcal{L}^s_{\chi^{(j+1)}} h \overset{(10)}{=} \sum_{s\geq 1} \frac{s}{(s+1)!} \mathcal{L}^s_{\chi^{(j+1)}} f^{(j)}, \tag{20}$$

one has $H^{(j+1)} = h + f^{(j+1)}$.

It is immediate from (13) that $\chi^{(j+1)}$ has the same null Taylor coefficients as $f^{(j)}$. Hence if $f^{(j)}$ is $(QxLy)$, then $\chi^{(j+1)}$ is $(QxLy)$ as well. It is easy to check by induction that this implies that $\mathcal{L}^s_{\chi^{(j+1)}} f^{(j)}$ is $(QxLy)$ for all s, then $f^{(j+1)}$ is $(QxLy)$. Similarly, Equation (15) implies that if $f^{(j)}$ is (Ly) then $\chi^{(j+1)}$ is $(QxLy)$ as well. This implies that $\mathcal{L}^s_{\chi^{(j+1)}} f^{(j)}$ is (Ly) for all s, hence $f^{(j+1)}$ is (Ly).

In order to show this, suppose by induction that $\mathcal{L}^s_{\chi^{(j+1)}} f^{(j)}$ is (Ly). Note that this is true for $s = 1$, as $\mathcal{L}_{\chi^{(j+1)}} f^{(j)} = \{y \cdot g^{(j)}, y \cdot C^{(j)}\} = y \cdot \sum_{l=1}^{n} (C_l^{(j)} \partial_{x_l} g^{(j)} - g_l^{(j)} \partial_{x_l} C^{(j)})$ is (Ly). Hence, recalling that $\mathcal{L}^{s+1}_{\chi^{(j+1)}} = \mathcal{L}_{\chi^{(j+1)}} \mathcal{L}^s_{\chi^{(j+1)}}$, write $\mathcal{L}^s_{\chi^{(j+1)}} f^{(j)} =: y \cdot F^{(j)}(x,t)$ and proceed similarly with $F^{(j)}$ in place of $g^{(j)}$.

This completes the formal part. In particular, by induction, $f^{(j)}$ is (Ly) for all j, then also is $\chi^{(j+1)}$, as claimed in Remark 1.3.

Let us now discuss the quantitative estimate on $f^{(j+1)}$ in the case $a > 0$. By Propositions 2.2 (set $f \leftarrow f^{(j)}$ and $\chi \leftarrow \chi^{(j+1)}$), 2.3 and the inductive hypothesis, one gets

$$\left\| \mathcal{L}^s_{\chi^{(j+1)}} f^{(j)} \right\|_{(1-2d_j)R_j} \leq s! \Theta^s \epsilon_j e^{-at}, \qquad \Theta := \frac{e^2 C_1}{a R_*^2 d_j^{2n+4}} \epsilon_j. \tag{21}$$

Setting $K := 2ne^2 C_1 R_*^{-2}$ and $\sigma := 2n + 5$, we have that

$$2n\Theta = (K \epsilon_j a^{-1} d_j^{-\sigma}) d_j \leq d_j, \tag{22}$$

as $\epsilon_{j+1}/\epsilon_j < 1$ by Lemma 3.1. Hence, $\Theta < 1/2$ and the series defined in (20) is convergent, furthermore

$$e^{at} \left\| f^{(j+1)} \right\|_{R_{j+1}} \leq \epsilon_j \sum_{s \geq 1} \Theta^s \leq 2n\Theta \epsilon_j \overset{(22)}{\leq} K a^{-1} d_j^{-\sigma} \epsilon_j^2 \overset{(18)}{=} \epsilon_{j+1}, \tag{23}$$

which completes the inductive step. The condition (17) in this case reads as

$$\epsilon_0 \leq a R_0^2 (2\pi)^{-\sigma} (8ne^2 C_1)^{-1}. \tag{24}$$

On the other hand, from the analyticity of f, we get $|f_{\alpha,\beta}(t)| \leq M_f R^{-|\alpha+\beta|} \leq M_f R_0^{-|\alpha+\beta|/16}$, as $R_0 \leq R^{16}$ by hypothesis. By using the first of (9) we get $\|f\|_{R_0} \leq M_f \sum_{(\alpha,\beta) \in \mathbb{N}^{2n}} R_0^{(15/16)|\alpha+\beta|} \leq 2ne^{(2n-1)} M_f R_0^{135/64} =: \epsilon_0$. Replacing the latter in (24), the condition on R_0 described in the statement of Theorem 1.1 is meant to be completed with the following one

$$R_0 \leq [a/(16(2\pi)^\sigma e^{2n+1} n^2 C_1 M_f)]^{64/7}. \tag{25}$$

The case $a = 0$ is analogous: it is sufficient to replace C_1 with C_2, remove the term $e^{\pm at}$ from the statements, (21) and (23), then replace a with 1 from (21) to (24), where now $\sigma = n + \tau + 5$. The only substantial difference consists in the sum obtained from (9), which is slightly improved, since f linear in y. We have in this case $\|f\|_{R_0} \leq n^2 e^{n-1} M_f R_0^{75/32} =: \epsilon_0$ leading to

$$R_0 \leq [8(2\pi)^\sigma e^{n+1} n^3 C_2 M_f]^{-32/11}. \tag{26}$$

□

C. Bounds on the coordinate transformation

Lemma 3.4. The transformation of coordinates defined by the limit (6) satisfies

$$|x^{(\infty)} - x|, |y^{(\infty)} - y|, |\eta^{(\infty)} - \eta| \leq R_0/6, \tag{27}$$

in particular, it defines an analytic map $\mathcal{M} : \mathcal{D}_{R_*} \to \mathcal{D}_{R_0}$ and $H^{(\infty)} := \mathcal{M}H$ is an analytic function on $\mathcal{D}_{R_*}$.

Proof. We will discuss the case $a > 0$. The case $a = 0$ is straightforward simply replacing C_1 with C_2, a with 1 and changing the value of σ, where necessary.

Let us start from the variable x. Note that, by Proposition 2.3, one has $\left\|\mathcal{L}^s_{\chi^{(j+1)}} x_l^{(j+1)}\right\|_{(1-2d_j)R_j} \leq s!\Theta^s R_0$ for all $l = 1,\ldots,n$. Hence we have, by (22),

$$|x^{(j+1)} - x^{(j)}| \leq n \max_{l=1,\ldots,n} \sum_{s\geq 1} \frac{1}{s!} \left\|\mathcal{L}^s_{\chi^{(j+1)}} x_l^{(j+1)}\right\|_{(1-2d_j)R_j} \leq 2nR_0\Theta \leq R_0 d_j.$$

In this way $|x^{(\infty)} - x| \leq \sum_{j\geq 0} |x^{(j+1)} - x^{(j)}|$ converges by (19). The procedure for y is analogous.

As for the third of (27), it is necessary to observe that $\mathcal{L}_{\chi^{(j+1)}}\eta = -\partial_t \chi^{(j+1)}$. Hence, by (16) and the second of (11), one has $\left\|\mathcal{L}^s_{\chi^{(j+1)}}\eta\right\|_{(1-2R_j)} \leq e^{-2} s!\Theta^{s-1}(R_*^2 e^{-2}\Theta) \leq s!\Theta^s R_0$, hence $|\eta^{(j+1)} - \eta^{(j)}| \leq 2nR_0\Theta \leq R_0 d_j$.

The bounds (27) ensure that points in $\mathcal{D}_{R_*}$ are mapped within $\mathcal{D}_{R_0}$ where $R_* = R_0/2$. Furthermore, the absolute convergence of the above described series, ensured by (19), guarantees the uniform convergence in every compact subset of $\mathcal{D}_{R_*}$ and the analyticity of $\mathcal{M}$, and then of $H^{(\infty)}$, follows from the theorem of Weierstraß, see, e.g., Ref. 4. □

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APPENDIX: PROOF OF PROPOSITION 2.1

First of all, recall $\sum_{|\nu|\geq N} |\nu|^\mu \mathcal{R}^{|\nu|} = \sum_{l\geq N} \binom{l+m-1}{m-1} l^\mu \mathcal{R}^l$. Now note that $\log \prod_{j=1}^{m-1}(l+j) \leq \int_1^m \log(l+x)dx = 1 - m + \log[(m+l)^{(m+l)}(1+l)^{-(1+l)}]$, hence $(m-1)!\binom{l+m-1}{m-1} = \prod_{j=1}^{m-1}(l+j) \leq e^{m-1}(m+l)^{(m+l)}(1+l)^{-(1+l)} \leq e^{2m-2}(m+l)^{(m+\mu)}$. This yields

$$\sum_{|\nu|\geq N} |\nu|^\mu \mathcal{R}^{|\nu|} \leq [e^{2m-2}/(m-1)!] \sum_{l\geq N} (m+l)^{(m+\mu)} \mathcal{R}^l. \tag{A1}$$

On the other hand, the function $h(x) := (m+x)^\kappa \mathcal{R}^{x/4}$ has a maximum in $x = 0$ (in the non-negative semi-axis) if $\mathcal{R} \leq \exp(-4\kappa/m)$ and in $x^* := -m - 4\kappa/\log\mathcal{R}$ otherwise. Hence, from (A1) with $\mu = 0$ we have $\sum_{|\nu|\geq N} \mathcal{R}^{|\nu|} \leq [(m-1)!]^{-1} m^m e^{2m-2} \sum_{l\geq N} \mathcal{R}^{(3/4)l}$ which gives the first of (9) by using the inequality $m^m \leq e^{m-1} m!$ and recalling $\mathcal{R} \leq e^{-4}$.

Now set $\mathcal{R} = 1 - \delta$. By hypothesis $\mathcal{R} > e^{-4}$, hence $(m+l)^{(m+\mu)}(1-\delta)^{l/4} \leq (1-\delta)^{-m/2}(-2(m+\mu)/\log(1-\delta))^{(m+\mu)}$. By substituting the latter in (A1) with $N = 0$, then using the inequalities $-\log(1-\delta) \geq \delta$ and $[1-(1-\delta)^{3/4}] \geq \delta/2$ as $\delta \leq 1/2$, the second of (9) easily follows.

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